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ON THE BEHAVIOR OF r- AND ϑ-CRACKS IN COMPOSITE MATERIALS UNDER THERMAL AND MECHANICAL LOADING

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Abstract—A mathematical study of radial matrix and interface cracking in the transverse direction of fiber-reinforced composites is presented. Two basic situations are considered: a radial crack in front of a circular fiber in an infinite matrix and an arc-shaped crack at the fiber/matrix interface. The first case is treated numerically using Erdogan's integral equation technique, whereas the second case allows for an analytical solution on the basis of the complex function method as developed by Muskhelishvili–Kolosov. In both cases the stress intensity factors are calculated for the full range of Dundurs' parameters.

1. INTRODUCTION

It is known that the strength of metals and the toughness of ceramics as well as other mechanical properties of ductile or brittle materials can greatly be improved by the addition of fibrous reinforcements. However, the transverse properties of such fiber-reinforced composites (FRCs) are often still an order of magnitude below the axial ones.

Thermal mismatch between the fiber and the surrounding matrix can lead to further degradation of the mechanical properties of the composite. During cooling from elevated processing temperatures down to room temperatures high thermal stresses develop in the vicinity of the fiber/matrix interface. If the stresses are tensile they will lead to matrix cracking or interface separation; if they are compressive they exert a stabilizing pressure on the fiber/matrix interface or in the matrix, unless they are counterbalanced and annihilated by an external tension.

This paper presents a mathematical study of radial matrix and interface cracking in transverse direction under the influence of thermal *and* mechanical loads. The following two basic situations are considered: a radial matrix crack (r-crack) in front of a single, circular fiber in an infinite matrix, and an arc-shaped crack (ϑ-crack) at the fiber/matrix interface. In both cases the thermal expansion and the elastic coefficients of the fiber and the matrix are assumed to be different and the matrix is loaded at infinity transversely to the fiber. The r-crack is simulated by a continuous distribution of dislocations which can be determined from a numerical solution of singular integral equations using an approach suggested by Erdogan *et al.* (1973, 1974, 1975). This distribution allows us to calculate the stress intensity factors at both crack tips. The ϑ-crack is analyzed using the complex function technique developed by Muskhelishvili–Kolosov (1953).

As has been shown by Müller *et al.* (in preparation (a)), the interaction between neighboring fibers and its influence on the stress distribution of the system can be neglected for fiber volume fractions up to 40%. Thus a single fiber model should describe r and ϑ-cracking reliably in FRCs with low or medium fiber volume content. Finally it should be mentioned that plasticity has not been taken into account: matrix as well as interface cracking is assumed to occur within the elastic deformation regime.

2. CALCULATION OF STRESS INTENSITY FACTORS OF r-CRACKS

2.1. *Formulation of the problem*

Consider the plane elastic system shown in Fig. 1. An elastic matrix, to which a uniaxial tension $\sigma_{22}^{\infty} = \sigma$ is applied at infinity, contains a single fiber of radius R. The elastic constants of the matrix and the fiber are denoted by (μ_1, κ_1) and (μ_2, κ_2), respectively, where μ_i is the shear modulus and $\kappa_i = 3-4\nu_i$ (plane strain), and $\kappa_i = (3-\nu_i)/(1+\nu_i)$ (plane stress) is Muskhelishvili's constant, ν_i being Poisson's ratio, $i = 1, 2$. The corresponding thermal expansion coefficients are α_1 and α_2, respectively.

We consider a radial matrix crack in front of the fiber perpendicular to the external stress field σ_{22}^{∞}. This arrangement is one of the most dangerous situations possible and we shall restrict ourselves to this case from now on.

In a series of papers Erdogan *et al.* (1973, 1974, 1975) have explained how to treat such problems mathematically. They simulate the crack by a continuous but unknown array of edge dislocations $f(t)$, which can be determined from the fact that the flanks of a crack must be free of forces. Since each dislocation leads to stresses along the flanks of the fictitious crack, $f(t)$ must be chosen such that the stresses from these contributions counterbalance the external loads in the undamaged material.

The mathematical analysis of this problem leads to a singular integral equation for the unknown distribution $f(t)$, which, in general, must be solved numerically.

2.2. *The integral equation*

The integral equation of the above-mentioned problem reads:

$$\int_a^b \frac{f(t)}{t-x}\,\mathrm{d}t + \int_a^b \{k_s(x,t) + k_f(x,t)\} f(t)\,\mathrm{d}t = -\frac{\pi(1+\kappa_1)}{2\mu_1} p(x), a \leqslant x \leqslant b. \quad (1)$$

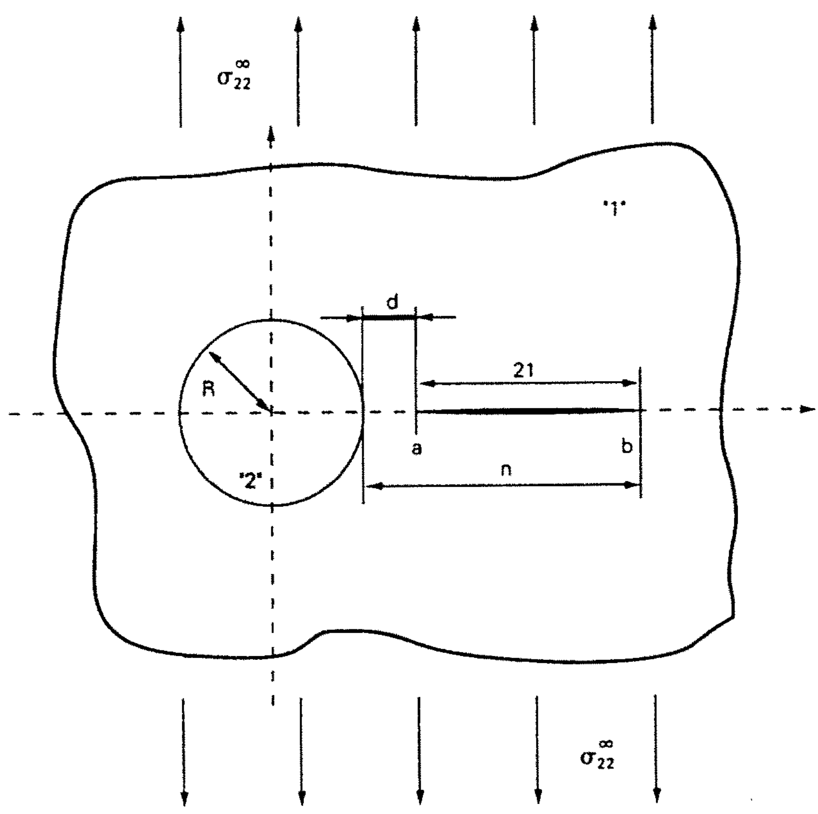

Fig. 1. Geometry of an r-crack near a fiber–matrix interface.

For a detailed derivation of this equation the reader is referred to Erdogan *et al.* (1974, 1975). Here we shall only summarize the results.

k_s and k_f denote the following integral kernels:

$$k_s(x,t) = \frac{1}{t-s}\left\{-2\frac{\alpha+\beta^2}{1-\beta^2}\frac{s}{2x}+\frac{\beta-\alpha}{1+\beta}\frac{1}{x^2}(3s^2-R^2)^2\left(1-\frac{2s}{t}\right)\right\}$$
$$+\frac{\beta-\alpha}{1+\beta}\left\{\left(1-\frac{4s}{t}\right)\frac{s(s^2-R^2)}{x^2(t-s)^2}-\frac{s^3(s^2-R^2)^2}{R^4t(t-s)^3}\right\},$$
$$k_f(x,t) = \frac{\beta-\alpha}{1+\beta}\frac{R^2}{x^2}\left(\frac{1}{2t}-\frac{3R^2}{tx^2}\right)-\left\{\frac{1-\alpha^2}{(1-\beta)(1+\alpha-2\beta)}-1\right\}\frac{R^2}{2tx^2}. \tag{2}$$

where the following contractions have been used:

$$s=\frac{R^2}{x},\quad m=\frac{\mu_2}{\mu_1},\quad \alpha=\frac{m(\kappa_1+1)-(\kappa_2+1)}{m(\kappa_1+1)+(\kappa_2+1)},\quad \beta=\frac{m(\kappa_1-1)-(\kappa_2-1)}{m(\kappa_1+1)+(\kappa_2+1)}. \tag{3}$$

α and β are Dundurs' parameters (Dundurs, 1969), which have proven to be extremely useful for the characterization of elastically mismatched composites. Note that k_s may become singular if the "a"-tip of the crack ends at the fiber, whereas k_f remains finite under these conditions.

In order to provide a unique solution of eqn (1) it is necessary to impose an additional condition which, physically speaking, is the continuity of displacements:

$$\int_a^b f(t)\,dt = 0. \tag{4}$$

The right-hand side of (1) contains the distribution of forces alongside the crack flanks in the undamaged material. We obtain for uniaxial mechanical and thermal loading (Muskhelishvili, 1953; Müller *et al.*, in preparation (a)):

$$p(x) = \sigma\left\{1-\frac{R^2}{x^2}\frac{\beta}{1+\alpha-2\beta}-\frac{3}{2}\frac{R^4}{x^4}\frac{\alpha-\beta}{1+\beta}+\frac{R^2}{x^2}\rho\frac{2(1+\alpha)}{1+\alpha-2\beta}\right\}. \tag{5}$$

where ρ is given by:

$$\rho=\begin{cases}\dfrac{2\mu_1/(1+\kappa_1)}{\sigma}[(1+\nu_2)\alpha_2-(1+\nu_1)\alpha_1](T-T_R), & \text{plane strain}\\[2ex] \dfrac{2\mu_1/(1+\kappa_1)}{\sigma}[\alpha_2-\alpha_1](T-T_R), & \text{plane stress.}\end{cases} \tag{6}$$

T_R denotes the fabrication temperature of the composite. Note that ρ is a measure of the relative strength of the thermal stresses when compared to the external mechanical loads. Negative values of ρ correspond to a composite in which the thermal expansion coefficient of the fiber is greater than that of the matrix. Consequently, negative values of ρ characterize compressive tangential stresses on the surface of the r-crack and vice versa.

2.3. *Stress intensity factors*

In order to calculate stress intensity factors, $f(t)$ is separated into a singular and into a non-singular part, the latter of which is called $F(t)$:

$$f(t) = F(t)(b-t)^{-1/2}(t-a)^{-1/2}. \tag{7}$$

Note that a separation like this holds only for the case of cracks which do not terminate at the interface. The equations for a terminating crack have been discussed by Müller *et al.* (In preparation (b)).

Consequently, with the definitions (Erdogan and Gupta, 1975):

$$\begin{aligned} K_I(b) &= -\frac{2\mu_1}{1+\kappa_1} 2^{1/2} \lim_{x\to b} [b-x]^{1/2} f(x), \\ K_I(a) &= \frac{2\mu_1}{1+\kappa_1} 2^{1/2} \lim_{x\to a} [x-a]^{1/2} f(x), \end{aligned} \tag{8}$$

the following expressions can be derived for the SIFs of r-cracks which do not terminate at the interface:

$$\begin{aligned} K_I(b) &= -\frac{2\mu_1}{1+\kappa_1}\sqrt{\frac{2\pi}{b-a}}F(b), \\ K_I(a) &= \frac{2\mu_1}{1+\kappa_1}\sqrt{\frac{2\pi}{b-a}}F(a). \end{aligned} \tag{9}$$

2.4. *Numerical solution of the integral equation*

We introduce the following dimensionless central crack coordinates $(\bar{x}, \bar{t})$:

$$x = l\bar{x} + L, \qquad t = l\bar{t} + L, \tag{10}$$

where

$$\bar{x} = x - L, \qquad \bar{t} = t - L, \qquad L = \frac{b+a}{2}, \qquad l = \frac{b-a}{2}. \tag{11}$$

With the definitions:

$$\begin{aligned} \bar{k}_s(\bar{x}, \bar{t}) &= l k_s(\bar{x}, \bar{t}), \qquad \bar{k}_f(\bar{x}, \bar{t}) = l k_f(\bar{x}, \bar{t}), \\ \bar{p}(\bar{x}) &= \frac{p(\bar{x})}{\Sigma}, \\ h(\bar{t}) &= \frac{2\mu_1}{1+\kappa_1} F(\bar{t}) \frac{1}{l\Sigma}, \\ w(\bar{t}) &= (1-\bar{t})^{-1/2}(\bar{t}+1)^{-1/2}, \end{aligned} \tag{12}$$

$$\Sigma = \begin{cases} \sigma & \text{if } \sigma \neq 0, \\ \left.\begin{aligned} &\frac{2\mu_1}{\kappa_1+1}[(1+\nu_2)\alpha_2 - (1+\nu_1)\alpha_1](T-T_R), && \text{plane strain} \\ &\frac{2\mu_1}{\kappa_1+1}[\alpha_2 - \alpha_1](T-T_R), && \text{plane stress} \end{aligned}\right\}, & \text{if } \sigma = 0, \end{cases}$$

the following dimensionless integral equation is obtained:

$$\int_{-1}^{+1} \frac{h(\bar{t})w(\bar{t})}{\bar{t}-\bar{x}}\,d\bar{t} + \int_{-1}^{+1} \{\bar{K}_s(\bar{x},\bar{t}) + \bar{K}_f(\bar{x},\bar{t})\} h(\bar{t})w(\bar{t})\,d\bar{t} = -\pi\bar{p}(\bar{x}), \qquad \int_{-1}^{+1} h(\bar{t})w(\bar{t})\,d\bar{t} = 0. \tag{13}$$

Following the Gauss–Jacobi integration technique of Erdogan *et al.* (1973) this equation can be mapped onto a system of linear equations for the unknown function $h(t)$ which is evaluated at the zeros of Chebyshev polynomials:

$$\sum_{k=1}^{N} h(\bar{t}_k) W_k \left\{ \frac{1}{\bar{t}_k - \bar{x}_j} + \bar{K}_s(\bar{x}_j, \bar{t}_k) + \bar{K}_f(\bar{x}_j, \bar{t}_k) \right\} = -\pi \bar{p}(\bar{x}_j), \qquad j = 1, \ldots, N-1,$$

$$\sum_{k=1}^{N} h(\bar{t}_k) W_k = 0, \tag{14}$$

with t_k, x_j from:

$$\bar{t}_k = \cos\left(\pi \frac{2k-1}{2N}\right), \qquad k = 1, \ldots, N, \tag{15}$$

$$\bar{x}_j = \cos\left(\pi \frac{j}{N}\right), \qquad j = 1, \ldots, N-1. \tag{16}$$

Thus for an r-crack which does not terminate at the interface, the SIFs of eqn (9) can be approximated by:

$$\frac{K_I(b)}{\Sigma\sqrt{\pi R}} = -h(\bar{t}_1)\sqrt{\frac{l}{R}}, \qquad \frac{K_I(a)}{\Sigma\sqrt{\pi l}} = h(\bar{t}_N)\sqrt{\frac{l}{R}}. \tag{17}$$

Note that as in the paper by Lu *et al.* (1990) the results are normalized with respect to the particle radius R and not with respect to the crack length l. This method of normalization has proven to be especially effective since the particle size is normally kept constant.

2.5. *Results and discussion*

This section presents some numerical data for the SIFs of r-cracks which do not terminate at the interface and which are either subject to purely thermal stresses (Section 2.5.1) or to a combination of thermal loading and uniaxial tension at infinity (Section 2.5.2). For further loading combinations and an extensive discussion of the case of a crack terminating at the interface, see the paper by Müller *et al.* (in preparation (b)).

2.5.1. *Thermal stresses.* Figure 2a–c shows a systematic study of the influence of Dundurs' parameters α and β on the SIFs of an r-crack of length $l/R = 1$. As expected, the stress intensity $K(b)$ at the remote crack tip "b" decreases monotonically as a function of increasing distance d/R. $K(b)$ increases slightly in magnitude for increasing Dundurs' parameters. However, the stress intensities $K(a)$ at crack tip "a" show a behavior which is less uniform. For distances d/R where the a-tip comes very close to the fiber ($0.001 < d/R < 0.1$) the normalized SIFs increase for positive α-values with increasing d/R, while they decrease for negative α. The curves for positive α show a maximum around $d/R = 0.1$. Beyond $d/R = 0.1$ *all* curves decrease monotonically, their slopes being a function of α and β. Increasing β-values lead to a slight increase of the maximum.

2.5.2. *Combined thermal and uniaxial mechanical loading.* A systematic study of crack shielding due to thermal stresses is shown in the sequence of Fig. 3 for different values of Dundurs' parameters. Depending upon the compressive ($\rho < 0$) or tensile nature ($\rho > 0$)

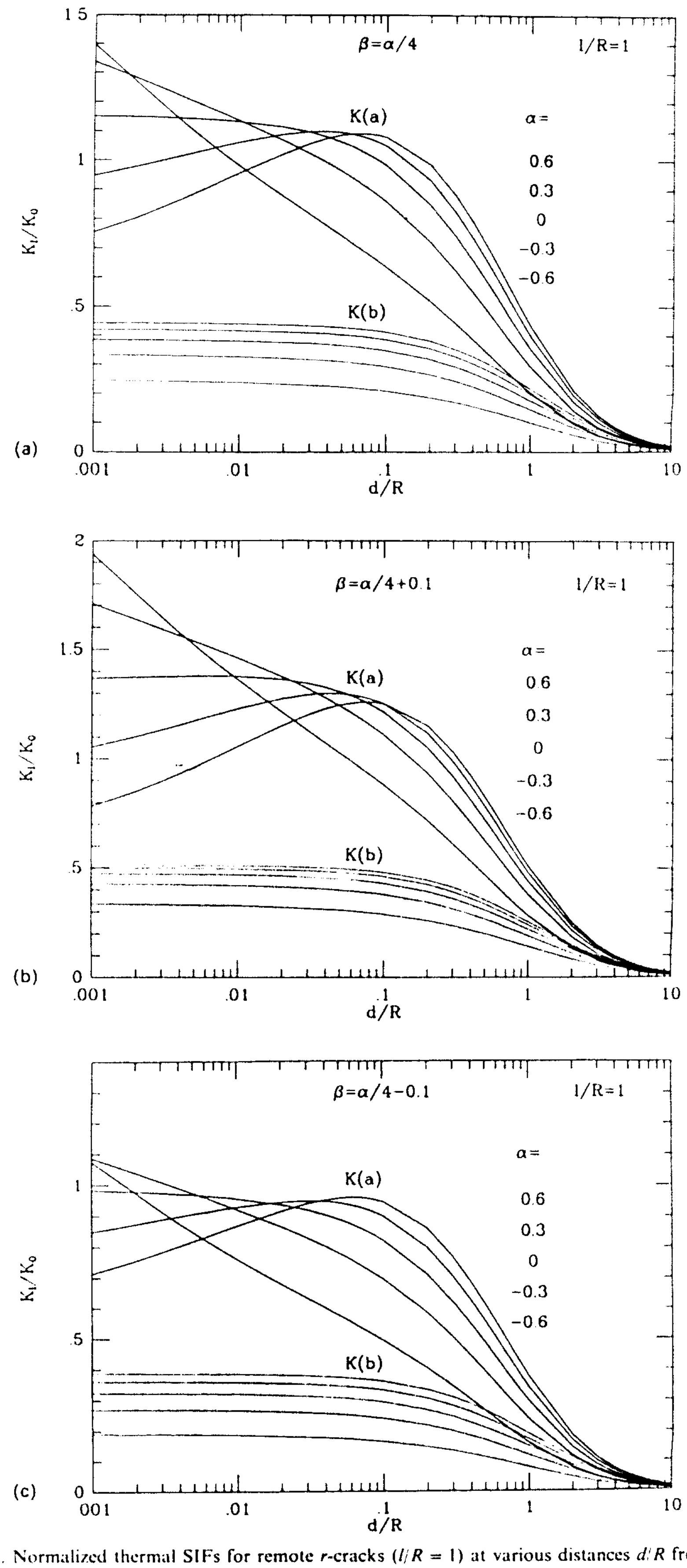

Fig. 2. Normalized thermal SIFs for remote r-cracks ($l/R = 1$) at various distances d/R from the interface and for different Dundurs' parameters.

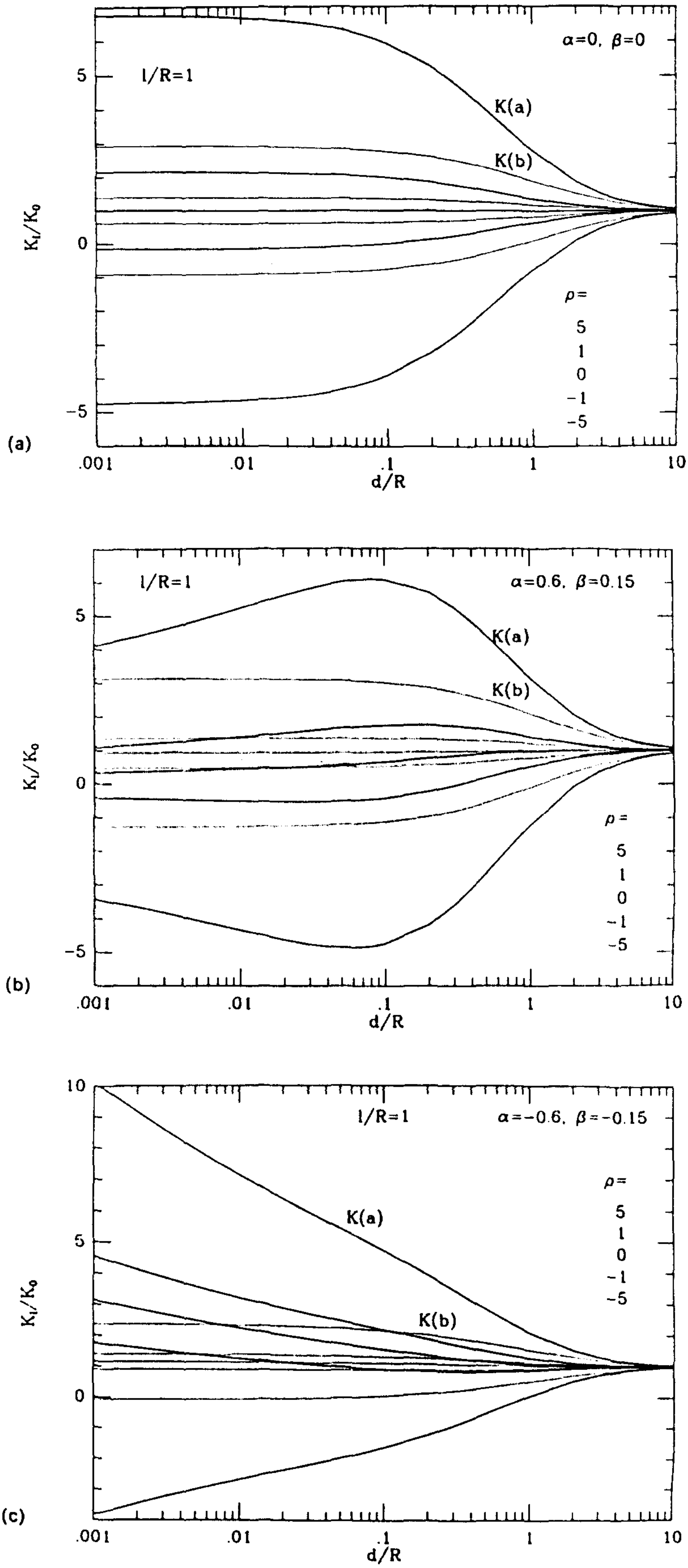

Fig. 3. Normalized SIFs for r-cracks under combined loading at various distances d/R ($l/R = 1$).

of the thermal stresses, the SIFs either decrease or increase when compared to the case of vanishing thermal mismatch ($\rho = 0$).

3. CALCULATION OF STRESS INTENSITY FACTORS OF ϑ-CRACKS

3.1. *Description of the problem*

Consider the plane elastic system shown in Fig. 4. An elastic matrix, to which biaxial tensions N_1, N_2 are applied at infinity, contains a single, circular fiber of radius R. The angle of inclination between N_1 and the real x-axis is called δ. Along the interface between the fiber and the matrix a circular-arc-shaped crack is oriented such that it subtends an angle of 2ζ symmetrically with respect to the positive x-axis. Furthermore, the crack is subject to an internal pressure p.

3.2. *The boundary conditions*

The followed first set of boundary conditions holds along the cracked part L_s and the bonded part L_b of the interface, respectively:

$$\left.\begin{aligned}\sigma^1_{rr}+i\sigma^1_{r\vartheta} &= -p\\ \sigma^2_{rr}+i\sigma^2_{r\vartheta} &= -p\end{aligned}\right\}\quad \text{on } L_s,$$

$$\left.\begin{aligned}\sigma^1_{rr}+i\sigma^1_{r\vartheta} &= \sigma^2_{rr}+i\sigma^2_{r\vartheta}\\ u^1_r+iu^1_\vartheta &= u^2_r+iu^2_\vartheta\end{aligned}\right\}\quad \text{on } L_b. \tag{18}$$

$\sigma^i_{rr}, \sigma^i_{r\vartheta}$, $i = 1, 2$ are the stress components in polar coordinates of the matrix region 1 and the region covered by the circular inclusion 2, respectively. u^i_r and u^i_ϑ, $i = 1, 2$, denote the corresponding displacements.

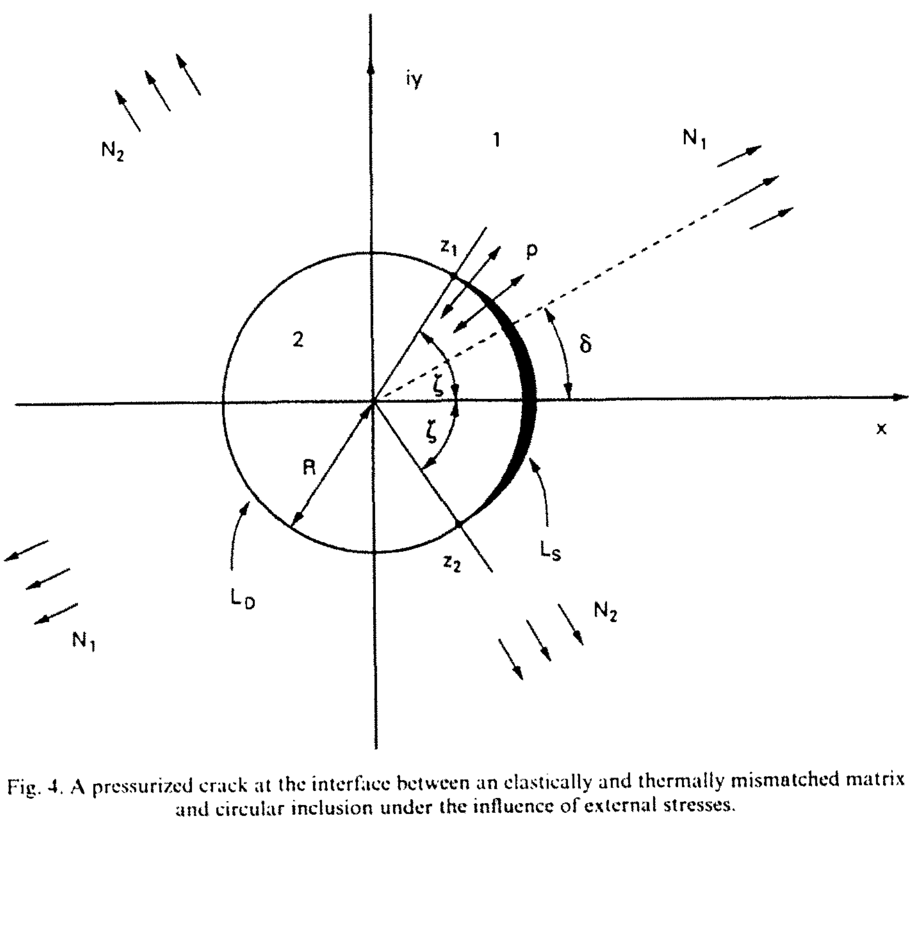

Fig. 4. A pressurized crack at the interface between an elastically and thermally mismatched matrix and circular inclusion under the influence of external stresses.

Furthermore, we demand that there is no rotation at infinity, that all stresses remain finite as $|z| \to 0$ and that all stresses tend towards N_1 and N_2 as $|z| \to \infty$. This forms the second set of boundary conditions.

3.3. *The Muskhelishvili–Kolosov equations for thermal stress problems*

Stresses and displacements in mechanically loaded, isothermal systems can be determined from the following set of generalized Muskhelishvili–Kolosov equations (Bogdanoff, 1954):

$$\sigma^i_{rr} + \sigma^i_{\vartheta\vartheta} = 4\{\Omega'_i(z) + \bar{\Omega}'_i(\bar{z})\}, \tag{19}$$

$$\sigma^i_{rr} + i\sigma^i_{r\vartheta} = 2\left\{\Omega'_i(z) + \bar{\Omega}'_i(\bar{z}) - \bar{z}\bar{\Omega}''_i(\bar{z}) - \frac{\bar{z}}{z}\bar{\omega}''_i(\bar{z})\right\}, \tag{20}$$

$$\mu_i(u^i_r + iu^i_\vartheta) = e^{-i\vartheta}\{\kappa_i\Omega_i(z) - z\bar{\Omega}'_i(\bar{z}) - \bar{\omega}'_i(\bar{z}) + \lambda_i z\}. \tag{21}$$

$\Omega_i(z)$ and $\omega'_i(z)$, $i = 1, 2$ are the well-known Goursat functions of two-dimensional elasticity (cf. e.g. Muskhelishvili, 1953) with:

$$\lambda_i = \mu_i\bar{\alpha}_i(T - T_R),$$
$$\bar{\alpha}_i = \begin{cases} (1+\nu_i)\alpha_i, & \text{plane strain,} \\ \alpha_i, & \text{plane stress.} \end{cases} \tag{22}$$

3.4. *Determination of the stress functions*

A close inspection (cf. Müller *et al.*, in preparation (c); or England, 1966, for details) of the first set of boundary conditions (18) shows that it is sufficient to determine only two complex functions $\Psi'(z)$ and $\Theta(z)$, which in terms of the four functions $\Omega_1(z)$, $\omega'_1(z)$, $\Omega_2(z)$ and $\omega'_2(z)$ are defined as follows:

$$\Theta(z) = \Omega_1(z) - z\bar{\Omega}'_2\left(\frac{R^2}{z}\right) - \bar{\omega}'_2\left(\frac{R^2}{z}\right), \quad z \in S_1,$$
$$\Theta(z) = \Omega_2(z) - z\bar{\Omega}'_1\left(\frac{R^2}{z}\right) - \bar{\omega}'_1\left(\frac{R^2}{z}\right), \quad z \in S_1,$$
$$\Psi'(z) = \mu_2\kappa_1\Omega_1(z) + \mu_1 z\bar{\Omega}'_2\left(\frac{R^2}{z}\right) + \mu_1\bar{\omega}'_2\left(\frac{R^2}{z}\right) + \mu_2\lambda_1 z - \nu\Theta'(z), \quad z \in 1,$$
$$\Psi'(z) = \mu_1\kappa_2\Omega_2(z) + \mu_2 z\bar{\Omega}'_1\left(\frac{R^2}{z}\right) + \mu_2\bar{\omega}'_1\left(\frac{R^2}{z}\right) + \mu_1\lambda_2 z - \nu\Theta'(z), \quad z \in 2, \tag{23}$$

where the following contractions were used:

$$\eta = \frac{1-\beta}{1+\beta}, \qquad \nu = -\frac{\mu_1(1-\eta\kappa_2)}{1+\eta}. \tag{24}$$

Furthermore, it can be shown (Müller *et al.*, in preparation (c); England, 1966) that these two functions must satisfy the following functional equations on L_s:

$$\Theta'^{+}(z) = \Theta'^{-}(z), \qquad \Psi'^{+}(z) + \eta\Psi'^{-}(z) = -\tfrac{1}{2}(\mu_1 + \mu_2\kappa_1)p + (\mu_2\lambda_1 + \eta\mu_1\lambda_2). \tag{25}$$

Note, that the indices "+" and "−" refer to the limit values of the corresponding functions approaching L_s from 1 and 2, respectively.

According to Muskhelishvili (1953) these equations have the following general solution:

$$\Theta'(z) = A_{-m}z^m + \cdots + A_{-1}z^1 + A_0 + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_n}{z^n},$$

$$\Psi'(z) = \left[-(\mu_1+\mu_2\kappa_1)\frac{p}{2} + (\mu_2\lambda_1+\eta\mu_1\lambda_2)\right]\frac{X_0(z)}{2\pi i}\int_L \frac{\mathrm{d}t}{X_0^+(t)(t-z)}$$

$$+X_0(z)\left\{\frac{B_n}{z^n} + \cdots + \frac{B_2}{z^2} + \frac{B_1}{z} + B_0 + B_{-1}z + \cdots + B_{-m}z^m\right\}. \tag{26}$$

where $X_0(z)$ denotes the Plemelj function:

$$X_0 = (z-R\mathrm{e}^{i\zeta})^{-1/2}(z-R\mathrm{e}^{-i\zeta})^{-1/2}\left(\frac{z-R\mathrm{e}^{i\zeta}}{z-R\mathrm{e}^{-i\zeta}}\right)^{i\gamma}, \qquad \gamma = \frac{1}{2\pi}\ln\eta. \tag{27}$$

The remaining constants $A_{-m}\ldots A_n$ and $B_{-m}\ldots B_n$ are determined from the second set of boundary conditions mentioned in Section 3.2 for the points at infinity and at the origin. An long and cumbersome calculation gives the following:

(1) if only thermal stresses and internal pressure are present:

$$\Theta'(z) = A_0,$$

$$\Psi'(z) = -\frac{\mu_1+\mu_2\kappa_1}{2}\frac{p(1-\beta)}{2}\{1-[z-R(\cos\zeta+2\gamma\sin\zeta)]X_0(z)\}$$

$$+\mu_1\mu_2\left(\bar{\alpha}_1+\frac{1+\beta}{1-\beta}\bar{\alpha}_2\right)(T-T_R)$$

$$+\mu_1\mu_2\frac{1+\beta}{2}(\bar{\alpha}_1-\bar{\alpha}_2)(T-T_R)[z-R(\cos\zeta+2\gamma\sin\zeta)]X_0(z)$$

$$-\mu_1(1+\kappa_2)\frac{1+\beta}{2}A_0[z-R(\cos\zeta+2\gamma\sin\zeta)]X_0(z),$$

$$p[1-(\cos\zeta+2\gamma\sin\zeta)\mathrm{e}^{-2\gamma\zeta}]-A_0$$

$$=\frac{1}{2}\frac{-\frac{1-\alpha}{1+\beta}\frac{2\mu_2}{1+\kappa_2}(\bar{\alpha}_1-\bar{\alpha}_2)(T-T_R)\left[1+\frac{1+\beta}{1-\beta}\mathrm{e}^{-2\gamma\zeta}(\cos\zeta+2\gamma\sin\zeta)\right]}{2\frac{1+\alpha}{1+\beta}+\frac{1-\alpha}{1+\beta}-\frac{1-\alpha}{1-\beta}\mathrm{e}^{-2\gamma\zeta}(\cos\zeta+2\gamma\sin\zeta)}, \tag{28}$$

where the Dundurs' material parameters α and β were used again;

(2) if only stresses at infinity are acting:

$$\Theta'(z) = a_1 + \frac{a_2}{z^2},$$

$$\Psi'(z) = \frac{(\mu_1+\mu_2\kappa_1)}{2}\frac{(1-\beta)}{2}\left\{b_{-1}[z-R(\cos\zeta+2\gamma\sin\zeta)]\right.$$

$$\left.-\frac{b_2}{z^2R}[z(\cos\zeta-2\gamma\sin\zeta)-R]\right\}X_0(z).$$

$$b_{-1} = \frac{\frac{1}{1-\beta}\frac{1+\alpha}{1-\alpha}(N_1+N_2) - \frac{1}{4}\frac{1+\alpha}{1-\beta}(N_1-N_2)[1+4\gamma^2]\sin^2\zeta\cos(2\delta)}{\frac{3+\alpha}{1-\alpha} - \frac{1+\beta}{1-\beta}e^{-2\gamma\zeta}(\cos\zeta+2\gamma\sin\zeta)}$$

$$+ i\frac{\frac{1}{4}\frac{1+\alpha}{1-\beta}(N_1-N_2)[1+4\gamma^2]\sin^2\zeta\sin(2\delta)}{1+\frac{1+\beta}{1-\beta}e^{-2\gamma\zeta}(\cos\zeta+2\gamma\sin\zeta)}.$$

$$b_2 = -\frac{R^3}{2}\frac{1+\alpha}{1+\beta}(N_1-N_2)e^{2\gamma\zeta}e^{2i\delta}.$$

$$a_1 = \frac{2}{1-\alpha}\frac{N_1+N_2}{4} - \frac{1-\beta}{2(1-\alpha)}b_{-1}.$$

$$a_2 = -\frac{R^2}{4}(N_1-N_2)e^{2i\delta}. \tag{29}$$

3.5. *Determination of the stress intensity factors*

Following Rice (1988) the SIFs of a ϑ-crack can be defined as follows:

$$K(Re^{i\zeta}) = \lim_{r\to 0} r^{i\gamma}\sqrt{2\pi r}\,(\sigma_{rr}-i\sigma_{r\vartheta}), \qquad K(Re^{-i\zeta}) = \overline{K(Re^{-i\zeta})}, \tag{30}$$

where r is a small radial distance at the interface in front of each crack tip, and σ_{rr} and $\sigma_{r\vartheta}$ denote the stresses directly at the interface, which in terms of the stress function are given by (Müller *et al.*, in preparation (c)):

$$\sigma_{rr}+i\sigma_{r\vartheta} = \frac{2}{\mu_1+\mu_2\kappa_1}\left[\frac{2}{1-\beta}\Psi'(z) - \mu_1\mu_2\left(\bar{\alpha}_1+\frac{1+\beta}{1-\beta}\bar{\alpha}_2\right)(T-T_R)\right]. \tag{31}$$

Hence, we obtain with eqns (28) and (29):

(1) for a pressurized ϑ-crack under the influence of residual thermal stresses:

$$K(Re^{i\zeta}) = \frac{2\frac{1-\alpha}{1-\beta}\left\{p\frac{1+\alpha-2\beta}{(1-\alpha)(1+\beta)}+\frac{\varepsilon_{th}}{1+\beta}\right\}}{\frac{3+\alpha}{1+\beta} - \frac{1-\alpha}{1-\beta}e^{-2\gamma\zeta}(\cos\zeta+2\gamma\sin\zeta)}(1-2i\gamma)\sqrt{\pi R\sin\zeta}\,e^{-\zeta\gamma}e^{i(\zeta/2+\gamma\ln(2R\sin\zeta))},$$

$$K(Re^{-i\zeta}) = \overline{K(Re^{i\zeta})}, \tag{32}$$

where ε_{th} characterizes the thermal mismatch:

$$\varepsilon_{th} = \frac{4\mu_2}{1+\kappa_2}(\bar{\alpha}_1-\bar{\alpha}_2)(T-T_R); \tag{33}$$

(2) for a ϑ-crack under the influence of stresses at infinity:

$$K(Re^{i\zeta}) = \{\bar{b}_{-1}-\bar{b}_2e^{i\zeta}\}(1-2i\gamma)\sqrt{\pi R\sin\zeta}\,e^{-\gamma\zeta}e^{i(\zeta/2+\gamma\ln(2R\sin\zeta))}, \qquad K(Re^{-i\zeta}) = \overline{K(Re^{i\zeta})}, \tag{34}$$

where b_{-1} and b_2 are specified in eqns $(29)_{3,4}$.

By superposition of both results the interaction between thermal and mechanical loads can now be studied. For a detailed discussion of special cases and the interaction see Müller *et al.* (in preparation (c)).

4. CONCLUSIONS

Stresses and stress intensity factors have been calculated for r- and ϑ-cracks in fiber-reinforced materials under thermal *and* mechanical loading. A numerical solution has been obtained for the r-cracks on the basis of singular integral equations whereas in the case of ϑ-cracks it was possible to obtain analytical results.

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